

PROOF100 — Foundations of Mathematical Reasoning & Computer Science

AUB Math Society & AUB Cryptography Club — Fall 2026

Problem Set 1: Logic and Deduction

Released: S3 — Monday 28 September 2026

Due: S5 — Monday 5 October 2026, at the start of the session

Covers: S1 (What a proof is), S2 (Propositional logic), S3 (Natural deduction)

Submission. On paper, in person, at the *start* of the session on which this is due. Handwritten or typeset are both fine; illegible is not. Write on one side, staple, and put your name on the first page. If you cannot attend that session, arrange to submit in advance by email.

Collaboration. Discussing problems with others is allowed and encouraged. Write up your own solutions in your own words, and name anyone you worked with at the top of your submission.

Tools. Using an LLM to explain a concept is fine. Using one to generate a solution you then submit is not. Any use at all must be disclosed in one line at the top of your submission.

Marking. Credit is given for stating the induction hypothesis explicitly, naming the technique you are using, and identifying where each hypothesis is used.

Problem 1. [22 points]

[Truth tables and the shape of a claim.] For the first two parts, a bare answer earns nothing: show the table and say what it shows.

- (a) [6 pts] Use a truth table to prove or disprove $\neg(P \vee (Q \wedge R)) \equiv (\neg P) \wedge (\neg Q \vee \neg R)$.
- (b) [6 pts] Use a truth table to prove or disprove $\neg(P \wedge (Q \vee R)) \equiv \neg P \vee (\neg Q \vee \neg R)$.
- (c) [4 pts] A classmate says: “The statement *every student who scored above 100 on the exam gets a prize* must be false, because nobody scored above 100.” Explain precisely why they are wrong, in terms of the truth table for $\rightarrow$.
- (d) [4 pts] Give a formula that is satisfiable but not a tautology, one that is a tautology, and one that is unsatisfiable, each using exactly the variables p and q . Justify each in one sentence.

Problem 2. [22 points]

[Equivalences and normal forms.]

- (a) [6 pts] Prove that $(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$ is a tautology *using logical equivalences only*.
- (b) [6 pts] Convert $\neg((p \wedge q) \rightarrow (r \vee \neg s))$ to conjunctive normal form (CNF).
- (c) [8 pts] The connective **nand** is defined by $P \mathbf{nand} Q \equiv \neg(P \wedge Q)$. Find expressions equivalent to $\neg A$, $A \wedge B$, $A \vee B$, and $A \rightarrow B$ using *only nand* and parentheses. You may use A and B as many times as you like.

Problem 3. [28 points]

[Natural deduction.] Give a derivation for each of the following. Write the derivation as a tree, label every rule application, and mark clearly where each assumption is discharged.

- (a) [5 pts] $p \wedge q \vdash q \wedge p$

- (b) [5 pts] $\vdash (p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$
- (c) [7 pts] $p \vee q, \neg p \vdash q$
- (d) [8 pts] $\vdash \neg(p \vee q) \rightarrow (\neg p \wedge \neg q)$

Problem 4 (find the bug). [8 points]

The derivation below concludes $q \rightarrow p$, $q \vdash p \rightarrow q$. The statement is true, but the derivation is not valid: exactly one of its two steps is wrong.

$$\frac{\frac{q \rightarrow p}{p} \quad q}{p \rightarrow q} \begin{array}{l} (1) \rightarrow E \\ (2) \rightarrow I, \text{ discharging } p \end{array}$$

Which step is wrong, (1) or (2)? In one or two sentences, say what that rule requires that is missing here.

Hint: $\rightarrow I$ may only discharge an assumption that was actually made.

Problem 5*. [20 points]

[Not everything is expressible.] Informally, a connective is *monotone* if flipping any input from false to true never flips the output from true to false. Both $\wedge$ and $\vee$ are monotone.

To make this precise, order the truth values by $F < T$. A formula $\varphi(p, q)$ in the two variables p and q is *monotone* if, for all valuations v and w ,

$$v(p) \leq w(p) \text{ and } v(q) \leq w(q) \text{ implies } v(\varphi) \leq w(\varphi).$$

In words: if w makes true every variable that v makes true, then w makes φ true whenever v does.

- (a) [6 pts] Generalise this definition to a formula $\varphi(p_1, \dots, p_n)$ in n variables, stated as a single quantified statement about two valuations v and w .
- (b) [8 pts] Prove by structural induction on formulas that every formula built using only $\wedge$, $\vee$, and propositional variables is monotone.
- (c) [6 pts] Conclude that $\{\wedge, \vee\}$ is not functionally complete by exhibiting a truth function that is not monotone, and therefore cannot be constructed by $\{\wedge, \vee\}$.

Video describing functional completeness: <https://www.youtube.com/watch?v=eZ2uURWN8TI>

You have not met structural induction formally yet (that is S9). For part (b), induct on the number of connectives in the formula and treat the base and two step cases separately.