Examples of Natural-Deduction Proofs (appendix for Lecture Slides 02)

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A propositional formula may be asserted semantically or proof-theoretically and, by soundness and completeness, the two assertions imply each other. In this set of notes, we illustrate the proof-theoretic approach with several small examples, starting with the well-known ***de Morgan's laws***. And we here choose one particular way of carrying out a formal proof: ***natural deduction***.

Within natural deduction, there are in fact several versions. The version used here is a little different from that in [LCHI, Chapter 2]. This is not to say that 'one is better than the other', but that there is a diversity of approaches in natural deduction, each with its own bookkeeping devices for manipulating formulas syntactically – and none is really best for all situations.

As pointed out in ***Lecture Slides 02***, if any of the rules in $\{(\text{LEM}), (\text{PBC}), (\neg\neg\text{E}), (\text{Peirce's})\}$ is added to the rules of IPC, we get ***classical propositional calculus*** (CPC).[1] In some examples of natural deduction in Sections 2, 3, 4, 5, and 6, we allow ourselves to use the preceding four rules of CPC. In Section 7, we comment on whether some of these formal proofs can be recast to be legal intuitionistically.

[1] These four are not the only possible rules we can add to IPC in order to get CPC, but they are perhaps the best known and sometime go by their Latin names: $(\text{LEM}) =$ *law of excluded middle* or *tertium non datur*, $(\text{PBC}) =$ *proof by contradiction* or *reductio ad absurdum*, $(\neg\neg\text{E}) =$ *elimination of double negation*, $(\text{Peirce's}) =$ *Peirce's law*.

1 de Morgan's Laws

de Morgan's laws can be asserted as four ***semantically valid formulas***:

1. $\models \neg(p \wedge q) \quad \to (\neg p \vee \neg q)$
2. $\models (\neg p \vee \neg q) \to \neg(p \wedge q)$
3. $\models \neg(p \vee q) \quad \to (\neg p \wedge \neg q)$
4. $\models (\neg p \wedge \neg q) \to \neg(p \vee q)$

Their semantic validity is established by using one of the formal semantics for the propositional calculus. For that purpose, we can try one of the semantics used in intuitionistic logic, *e.g.*, one based on Heyting algebras or one based on Kripke semantics (see [LCHI, Sections 2.4-2.5]); this will succeed with the second, the third, and the fourth, of ***de Morgan's laws***, but will not with the first (more on this in Section 7). For the first, we have to use a semantics that works for classical logic, *e.g.*, one based on Boolean algebras.

de Morgan's laws can also be asserted in the form of four ***formally deducible judgements***:

1. $\vdash \neg(p \wedge q) \quad \to (\neg p \vee \neg q)$
2. $\vdash (\neg p \vee \neg q) \to \neg(p \wedge q)$
3. $\vdash \neg(p \vee q) \quad \to (\neg p \wedge \neg q)$
4. $\vdash (\neg p \wedge \neg q) \to \neg(p \vee q)$

In following sections are natural-deduction proofs for these four laws.

Remark on notation and terminology. The symbol '$\vdash$' is used in slightly different (but related) ways by different authors. This is often confusing and ambiguous. I will use '$\vdash$' in two ways only. (And these two ways are not all the possibilities adopted by others.) Trying to reduce the overhead in a field replete with formal symbols, I will rely on the context to disambiguate the meaning of '$\vdash$', although it will often make little or no difference which of the two meanings is intended:

(a) Sometimes '$\vdash$' is a symbol at the ***meta level***, just like the symbol '$\models$', *i.e.*, it is not part of the syntax of the logic under consideration and is not affected by the formal apparatus to operate on it. In this sense, an expression of the form '$\Gamma \vdash \varphi$' means that the wff φ is formally deducible (or derivable) from the set Γ of *premises* (or, when we get to typed λ-calculi, *type assumptions*). In this sense, we can also write '$\Gamma \nvdash \varphi$' to mean it is not the case that '$\Gamma \vdash \varphi$'.

(b) Sometimes '$\vdash$' is a symbol at the ***formal level***, *i.e.*, it is part of the syntax of the logic. In this sense, the expression '$\Gamma \vdash \varphi$' is a formal object which is operated on by the formal apparatus of the logic. I will call '$\Gamma \vdash \varphi$' a *judgement* (and I will try to avoid the word *sequent* that some authors use in place of *judgement*). In this sense, '$\Gamma \nvdash \varphi$' is malformed.

Note that the way we organize a natural deduction, as a linearly-numbered sequence of wff's, the symbol '$\vdash$' is not used. However, after completing the deduction, we may write something like '$\Gamma \vdash \varphi$' which, in this case, can be taken in either of the two meanings.

2 Natural-Deduction Proof of de Morgan's Law (1)

The following is a natural-deduction formal proof for the first ***de Morgan's law***:

1	$\neg(p \wedge q)$	assume
2	$\neg(\neg p \vee \neg q)$	assume
3	$\neg p$	assume
4	$(\neg p \vee \neg q)$	$\vee$I 3
5	$\bot$	$\neg$E 2, 4
6	$\neg\neg p$	$\neg$I 3-5
7	$\neg q$	assume
8	$\neg p \vee \neg q$	$\vee$I 7
9	$\bot$	$\neg$E 2, 8
10	$\neg\neg q$	$\neg$I 7-9
11	p	$\neg\neg$E 6
12	q	$\neg\neg$E 10
13	$p \wedge q$	$\wedge$I 11, 12
14	$\bot$	$\neg$E 1, 13
15	$\neg\neg(\neg p \vee \neg q)$	$\neg$I 2-14
16	$(\neg p \vee \neg q)$	$\neg\neg$E 15
17	$\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$	$\rightarrow$I 1-16

3 Natural-Deduction Proof of de Morgan's Law (2)

The following is a natural-deduction formal proof for the second ***de Morgan's law***:

1	$\neg p \vee \neg q$	assume
2	$p \wedge q$	assume
3	p	$\wedge$E$_1$
4	q	$\wedge$E$_2$
5	$\neg p$	assume
6	$\neg q$	assume
7	p	assume
8	$\perp$	$\neg$E 4, 6
9	$\neg p$	$\neg$I 7-8
10	$\neg p$	$\vee$E 1, 5-5, 6-9
11	$\perp$	$\neg$E 3, 10
12	$\neg(p \wedge q)$	$\neg$I 2-11
13	$(\neg p \vee \neg q) \rightarrow \neg(p \wedge q)$	$\rightarrow$I 1-12

4 Natural-Deduction Proof of de Morgan's Law (3)

The following is a natural-deduction formal proof for the third ***de Morgan's law***:

1	$\neg(p \lor q)$	assume
2	p	assume
3	$p \lor q$	$\lor$I 2
4	$\bot$	$\neg$E 1, 3
5	$\neg p$	$\neg$I 2-4
6	q	assume
7	$p \lor q$	$\lor$I 6
8	$\bot$	$\neg$E 1, 7
9	$\neg q$	$\neg$I 6-8
10	$\neg p \land \neg q$	$\land$I 5, 9
11	$\neg(p \lor q) \to (\neg p \land \neg q)$	$\to$I 1-10

5 Natural-Deduction Proof of de Morgan's Law (4)

The following is a natural-deduction formal proof for the fourth ***de Morgan's law***:

1	$\neg p \wedge \neg q$	assume
2	$\neg p$	$\wedge$E 1
3	$\neg q$	$\wedge$E 1
4	$p \vee q$	assume
5	p	assume
6	q	assume
7	$\neg p$	assume
8	$\perp$	$\neg$E 3, 6
9	$\neg\neg p$	$\neg$I 7-8
10	p	$\neg\neg$E 9
11	p	$\vee$E 4, 5-5, 6-10
12	$\perp$	$\neg$E 2, 11
13	$\neg(p \vee q)$	$\neg$I 4-12
14	$(\neg p \wedge \neg q) \to \neg(p \vee q)$	$\to$I 1-13

6 Natural-Deduction Proof of de Morgan's Law (4) – again

A natural-deduction formal proof for the fourth ***de Morgan's law***, once more and differently:

1	$\neg p \wedge \neg q$			assume
2	$\neg p$			$\wedge\mathrm{E}_1$ 1
3	$\neg q$			$\wedge\mathrm{E}_2$ 1
4	$p \vee q$			assume
5	p	assume	q	assume
6			$\neg p$	assume
7			$\perp$	$\neg\mathrm{E}$ 3, 5
8			$\neg\neg p$	$\neg\mathrm{I}$ 6-7
9			p	$\neg\neg\mathrm{E}$ 8
10	p			$\vee\mathrm{E}$ 4, 5-5, 5-9
11	$\perp$			$\neg\mathrm{E}$ 2, 10
12	$\neg(p \vee q)$			$\neg\mathrm{I}$ 4-11
13	$(\neg p \wedge \neg q) \rightarrow \neg(p \vee q)$			$\rightarrow\mathrm{I}$ 1-12

7 Concluding Remarks on de Morgan's Laws

The natural deductions in Sections 3 and 4 do not use any rule in $\{(\text{LEM}), (\text{PBC}), (\neg\neg\text{E}), (\text{Peirce's})\}$ and are therefore intuitionistically legal. However, the other natural deductions besides those two in preceding sections are not intuitionistically legal, because:

- The formal proof in Section 2 uses $(\neg\neg\text{E})$ in line 11 and line 12.
- The formal proof in Section 5 uses $(\neg\neg\text{E})$ in line 10.
- The formal proof in Section 6 uses $(\neg\neg\text{E})$ in line 9.

Sections 5 and 6 are classical formal proofs for the same formula $(\neg p \wedge \neg q) \to \neg(p \vee q)$, which is the fourth of ***de Morgan's laws***. It turns out that, with a little effort, one can construct an alternative formal proof for the same formula which does not use any of the rules in $\{(\text{LEM}), (\text{PBC}), (\neg\neg\text{E}), (\text{Peirce's})\}$, which is therefore intuitionistically legal.

Exercise 1. Construct a natural-deduction proof for $(\neg p \wedge \neg q) \to \neg(p \vee q)$, the fourth of ***de Morgan's laws***, which is intuitionistically legal.

Exercise 2. Search the Web for an explanation for why any proof of the wff $\neg(p \wedge q) \to (\neg p \vee \neg q)$, the first of ***de Morgan's laws*** in Section 2, is bound not to be intuitionistically legal.

Hint: An impossibility result, such as this one, is best handled model-theoretically, *i.e.*, by invoking one of the standard formal semantics of intuitionistic logic. A good starting point is the webpage: Do De Morgan's laws hold in propositional intuitionistic logic? where an argument based on Heyting algebras and topological spaces is sketched. Can you come up with an argument based on Kripke models?

8 Other Small Examples of Natural-Deduction Proofs

8.1 Formal proof of the judgement $p \vdash q \to (p \wedge q)$

1	p	premise
2	q	assume
3	$p \wedge q$	$\wedge$I 1, 2
4	$q \to (p \wedge q)$	$\to$I

8.2 Formal proof of the judgement $p \to (q \to r) \vdash p \wedge q \to r$

1	$p \to (q \to r)$	premise
2	$p \wedge q$	assume
3	p	$\wedge E_1$ 2
4	$q \to r$	$\to$E 1, 3
5	q	$\wedge E_2$ 2
6	r	$\to$E 4, 5
7	$p \wedge q \to r$	$\to$I

8.3 Formal proof of the judgement $p \wedge q \to r \vdash p \to (q \to r)$

1	$p \wedge q \to r$	premise
2	p	assume
3	q	assume
4	$p \wedge q$	$\wedge$I 2, 3
5	r	$\to$E 1, 4
6	$q \to r$	$\to$I
7	$p \to (q \to r)$	$\to$I

8.4 Formal proof of the judgement $p \to (q \to r) \vdash (p \to q) \to (p \to r)$

1	$p \to (q \to r)$	premise
2	$p \to q$	assume
3	p	assume
4	q	$\to$E 2, 3
5	$q \to r$	$\to$E 1, 3
6	r	$\to$E 5, 4
7	$p \to r$	$\to$I
8	$(p \to q) \to (p \to r)$	$\to$I

8.5 Formal proof of the judgement $p \wedge \neg q \to r,\ \neg r,\ p \ \vdash \ q$

1	$p \wedge \neg q \to r$	premise
2	$\neg r$	premise
3	p	premise
4	$\neg q$	assume
5	$p \wedge \neg q$	$\wedge$I 3, 4
6	r	$\to$E 1, 5
7	$\bot$	$\neg$E 6, 2
8	$\neg\neg q$	$\neg$I
9	q	$\neg\neg$E 8

Exercise 3. The natural-deduction examples in Subsections 8.1, 8.2, 8.3, and 8.4, are all intuitionistically legal, but not the one in Subsection 8.5 which uses $(\neg\neg\text{E})$ in line 9. Can you recast this last natural deduction so that it becomes intuitionistically legal? If not, can you provide a semantic (*i.e.*, model-theoretic) argument why the judgement $p \wedge \neg q \to r,\ \neg r,\ p \ \vdash \ q$ cannot hold intuitionistically?

Hint: The judgement in Subsection 8.5 cannot be derived intuitionistically. A model-theoretic argument is a little harder. A little easier is to show that, if the judgement could be derived intuitionistically, then one of the rules disallowed in IPC (such as $(\neg\neg\text{E})$ or (PBC) or (LEM) or some other) would be derivable intuitionistically.