PROOF101, Spring 2026, Lecture Slides 02

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Intuitionistic Logic

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What Is Intuitionistic Logic?

for history and justification:

- read preamble in [LCHI, Chapt 2, pp 27-28]
- read on the Web the BHK interpretation of intuitionistic logic

closer to our concerns in this course, intuitionistic logic is:

- essential background for *Curry-Howard Isomorphism* –
  we have several correspondences, informally, among several others:

logic / proofs	*programming / computation*
formulas	types
proofs	programs
tautologies	types of closed λ-terms
normalization	computation

Outline of Lecture Slides 02

- most details in these lecture slides are from the book:

  Lectures on the Curry-Howard Isomorphim

  by Sorensen and Urzyczyn – henceforth denoted [LCHI]

- this set of slides mostly restricted to *intuitionistic propositional logic* / *intuitionistic propositional calculus* with a few comparisons with *classical propositional calculus*

- main headings in this set of slides :
  - *syntax of propositional logic* [LCHI, Sect 2.1]
  - *natural deduction* [LCHI, Sect 2.2]
  - *semantics of classical logic: Boolean algebras* [LCHI, Sect 2.3]
  - *semantics of intuitionistic logic: Heyting algebras* [LCHI, Sect 2.4]
  - *another semantics of intuitionistic logic: Kripke models* [LCHI, Sect 2.5]
  - *implicational fragment of intuitionistic logic* [LCHI, Sect 2.6]

Outline of Lecture Slides 02

Important things to remember from [LCHI, Chapter 2] :

- proof rules for *natural deduction*, slides 8 – 12 ,
- *Heyting algebras* vs. *Boolean algebras*, slides 14 – 15 ,
- *soundness* and *completeness* relative to Heyting algebras, slide 18 ,
- *soundness* and *completeness* relative to Kripke models, slide 20 ,
- reasons for studying the *implicational fragment of IPC*, slide 21 .

A Partial Glossary

Very quickly, we have to face many acronyms for naming different fragments of *propositional logic* and their natural-deduction *proof systems*. Here is a partial list: [1]

$$\mathrm{IPC}(\to,\bot,\wedge,\vee) = \mathrm{IPC} = \text{full } \textit{intuitionistic propositional calculus}$$
$$\mathrm{CPC}(\to,\bot,\wedge,\vee) = \mathrm{CPC} = \text{full } \textit{classical propositional calculus}$$

logic-fragment acronym	name description	proof-system acronym
$\mathrm{IPC}(\to)$	*implicational* fragment of IPC	$\mathrm{NJ}(\to)$
$\mathrm{IPC}(\to,\bot)$	*implicational w/* $\bot$ fragment of IPC	$\mathrm{NJ}(\to,\bot)$
$\mathrm{IPC}(\to,\wedge,\vee)$	*minimal* fragment of IPC	$\mathrm{NJ}(\to,\wedge,\vee)$
$\mathrm{IPC}(\to,\bot,\wedge,\vee)$	*full* IPC	$\mathrm{NJ}(\to,\bot,\wedge,\vee)$
$\mathrm{CPC}(\to)$	*implicational* fragment of CPC	$\mathrm{NK}(\to)$
$\mathrm{CPC}(\to,\bot)$	*implicational w/* $\bot$ fragment of CPC	$\mathrm{NK}(\to,\bot)$
$\mathrm{CPC}(\to,\wedge,\vee)$	*minimal* fragment of CPC	$\mathrm{NK}(\to,\wedge,\vee)$
$\mathrm{CPC}(\to,\bot,\wedge,\vee)$	*full* CPC	$\mathrm{NK}(\to,\bot,\wedge,\vee)$

[1] Some of these appear do not yet appear in [LCHI, Chapt 2] by only later in [LCHI] .

Syntax of IPC

Syntax of IPC = Syntax of CPC,
but their semantics are different and set them apart :

- a *well-formed formula* (wff) of propositional logic is
  - *atomic*: $\bot$ ('absurdity') or a propositional variable *p*, or
  - *composite*: $(\varphi \to \psi)$ or $(\varphi \wedge \psi)$ or $(\varphi \vee \psi)$,
    where φ and ψ are previously defined wff's .
- with Φ denoting the set of propositional wff's, an extended BNF:

  $$\varphi, \psi \in \Phi ::= \bot \mid p \mid \varphi \to \psi \mid \varphi \wedge \psi \mid \varphi \vee \psi$$

  where *p* ranges over an infinite set PV of propositional variables.
- *abbreviations* using the *logical connectives* in $\{\to, \bot, \wedge, \vee\}$:

  $$\begin{array}{lll} \neg\varphi & \triangleq & \varphi \to \bot \\ \varphi \leftrightarrow \psi & \triangleq & (\varphi \to \psi) \wedge (\psi \to \varphi) \\ \top & \triangleq & \bot \to \bot \ = \ \neg\bot \end{array}$$

  ('$\neg$' usually taken as a *connective* in CPC)

  conventions for omitting parentheses in [LCHI, 2.1.2, p 29] .

Natural Deduction: A Formal Proof System for IPC

- many different ways of setting up a *formal-proof system* for intuitionistic logic:
  - *Hilbert-style proof systems*,
  - *Gentzen's sequent calculi*,
  - *natural deduction*,
  - many variations on each of the preceding.
- *natural deduction* in [LCHI, Sect 2.2] is one possible way, the most '*natural*' to flesh out the Curry-Howard Isomorphism.
- *natural deduction* in these slides is a little different from [LCHI, Sect 2.2] :
  - here, a slight variation on bookkeeping in formal proofs,
  - closer to the presentation in [LCHI, Sect 4.4, pp 82-93] .

Natural Deduction: Proof Rules for IPC

introduction and ***elimination*** rules for each of $\{\wedge, \vee, \rightarrow\}$:

for $\wedge$:

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}\ (\wedge\mathbf{I}) \qquad\qquad \frac{\varphi \wedge \psi}{\varphi}\ (\wedge\mathbf{E}_1) \qquad\qquad \frac{\varphi \wedge \psi}{\psi}\ (\wedge\mathbf{E}_2)$$

for $\vee$:

$$\frac{\varphi}{\varphi \vee \psi}\ (\vee\mathbf{I}_1) \qquad\qquad \frac{\psi}{\varphi \vee \psi}\ (\vee\mathbf{I}_2) \qquad\qquad \frac{\varphi \vee \psi \quad \boxed{\begin{array}{c}\varphi\\ \vdots\\ \theta\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \theta\end{array}}}{\theta}\ (\vee\mathbf{E})$$

for $\rightarrow$:

$$\frac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \psi\end{array}}}{\varphi \rightarrow \psi}\ (\rightarrow\mathbf{I}) \qquad\qquad \frac{\varphi \quad \varphi \rightarrow \psi}{\psi}\ (\rightarrow\mathbf{E})$$

remark: in the rules with boxes, $(\rightarrow\mathrm{I})$ and $(\vee\mathrm{E})$, the ellipsis points '. . .' may be empty, thus in rule $(\rightarrow\mathrm{I})$ it is possible that $\varphi = \psi$ and similarly in rule $(\vee\mathrm{E})$.

Natural Deduction: Proof Rules for IPC

- rules for $\{\to, \wedge, \vee\}$ come in duals: ***introduction*** + ***elimination***
- for $\perp$, only an ***elimination*** rule, no ***introduction*** rule : [2]

$$\frac{\perp}{\varphi} \quad (\perp\mathbf{E})$$

(also called '*ex falso quodlibet*' or just '*ex falso*')

- for abbreviation $\top \triangleq \perp \to \perp$, only an ***introduction*** rule, no ***elimination*** rule :

$$\frac{}{\top} \quad (\top\mathbf{I})$$

Remark: When we later flesh out connections of the *Curry-Howard Isomorphism*, *proof rules* that come in duals (*introduction* + *elimination*) will correspond to *programming/computational rules* that come in duals (*constructor* + *destructor*):

	introduction / constructor	*elimination / destructor*
$\to$	λ	apply
$\wedge$	$\langle\ ,\ \rangle$	π_1, π_2
$\vee$	$\mathbf{in}_1, \mathbf{in}_2$	**case**

[2] Bertrand Russell: "Grant me a contradiction, and I'll prove that I'm the pope!"

Natural Deduction: Proof Rules for IPC

- ***derived rules***
  - for the abbreviation $\neg\varphi \triangleq \varphi \to \bot$, again two dual rules, **(¬I)** + **(¬E)**:

$$\dfrac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \bot\end{array}}}{\neg\varphi}\ (\neg\mathbf{I}) \qquad\qquad \dfrac{\varphi \qquad \neg\varphi}{\bot}\ (\neg\mathbf{E})$$

  - in the style of [LCHI, Figure 2.1, p 33] , we may write instead :

$$\dfrac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\varphi}\ (\neg\mathbf{I}) \qquad\qquad \dfrac{\Gamma \vdash \neg\varphi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \bot}\ (\neg\mathbf{E})$$

Natural Deduction: Other Proof Rules?

- In some accounts of natural deduction, the rule for *disjunction elimination* is given as $(\vee\mathbf{E}^*)$: [3]

$$\dfrac{\boxed{\begin{array}{c}\varphi\\ \vdots\\ \theta\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \theta\end{array}}}{(\varphi \vee \psi) \to \theta} \quad (\vee\mathbf{E}^*)$$

  which is often more convenient to use than the standard $(\vee\mathbf{E})$.

- **FACT:** *$(\vee\mathbf{E})$ and $(\vee\mathbf{E}^*)$ are inter-derivable intuitionistically:*
  1. *$(\vee\mathbf{E})$ is derivable from $(\vee\mathbf{E}^*)$ and $(\to\mathbf{E})$.*
  2. *$(\vee\mathbf{E}^*)$ is derivable from $(\vee\mathbf{E})$ and $(\to\mathbf{I})$.*

[3] We name it $(\vee\mathrm{E}^*)$ with a superscript '$*$' to distinguish it from the more standard rule $(\vee\mathrm{E})$.

Natural Deduction: Other Proof Rules?

- to obtain ***classical propositional logic / calculus*** (CPC)
  we can add any of the following four rules, none allowed in IPC:

$$\frac{}{\varphi \vee \neg\varphi}\ (\textbf{LEM}) \qquad \frac{\boxed{\begin{array}{c}\neg\varphi \\ \vdots \\ \bot\end{array}}}{\varphi}\ (\textbf{PBC})$$

$$\frac{\neg\neg\varphi}{\varphi}\ (\neg\neg\textbf{E}) \qquad \frac{}{((\varphi \to \psi) \to \varphi) \to \varphi}\ (\textbf{Peirce's})$$

(LEM) $=$ *law of excluded middle*, (PBC) $=$ *proof by contradiction*
($\neg\neg$E) $=$ *elimination of double negation*, (Peirce's) $=$ *Peirce's law*

- these four rules are inter-derivable intuitionistically ,
- adding any one of the four produces CPC .

Semantics of CPC: Boolean Algebras (BA's)

- The semantics of CPC are usually based on the *two-valued Boolean algebra* $\mathbb{B}$:
  - $\mathbb{B} \triangleq \langle \{\textit{false}, \textit{true}\}, \vee, \wedge, \textit{false}, \textit{true}, \neg \rangle$
  - or, more simply, $\mathbb{B} \triangleq \langle \{0, 1\}, \vee, \wedge, 0, 1, \neg \rangle$ with $0 = \textit{false}$ and $1 = \textit{true}$.
- $\mathbb{B}$ is only the simplest instance of a more general notion of *Boolean algebra*.
- Basic properties of Boolean algebras are reviewed in [LCHI, Sect 2.3].
- **THEOREM** (Stone's Representation Theorem):
  Every Boolean algebra is isomorphic to a field of sets [LCHI, Thm 2.3.9, p. 37].
- **THEOREM:** *A propositional wff φ is a classical tautology (i.e., $\mathbb{B} \models \varphi$) iff $\mathcal{B} \models \varphi$ for every BA $\mathcal{B}$* .
  - The proof is in [LCHI, Thm 2.3.11, p. 37].
  - *φ is a tautology = φ is semantically valid*,
    which can be conveniently certified by
    φ's truth-table where last-column entries are all true.

Semantics of IPC: Heyting Algebras versus BA's

1. A ***bounded distributive lattice*** [4] (BDL) is denoted $\mathcal{L} \triangleq \langle L, \sqcup, \sqcap, 0, 1 \rangle$ where the partial order $\leqslant$ is implicit and defined by: $a \leqslant b$ iff $a \sqcup b = b$
   - $a \in L$ has a *complement* $b \in L$ iff $a \sqcap b = 0$ and $a \sqcup b = 1$.

   FACT: *$a \in L$ may or may not have a complement, but if it does, its complement is unique, i.e. complements are uniquely defined in BDL's.*

2. A ***Heyting algebra*** (HA) is a BDL $\mathcal{H} \triangleq \langle H, \sqcup, \sqcap, 0, 1, \Rightarrow \rangle$ with an extra binary operation "$\Rightarrow$" satisfying the condition:

   $$a \leqslant (b \Rightarrow c) \quad \text{iff} \quad (a \sqcap b) \leqslant c \qquad \text{for all } a, b, c \in H$$

   somewhat mysterious condition, but compare with BA's to clarify . . .

3. A ***Boolean algebra*** (BA) is a BDL $\mathcal{B} \triangleq \langle B, \sqcup, \sqcap, 0, 1, - \rangle$ where
   - every $a \in B$ has a complement, and
   - "$-$" is the unary operator mapping a to its unique complement.

   FACT: *Every BA is a HA, by defining $(a \Rightarrow b) \triangleq (-a \sqcup b)$ for all $a, b \in B$.*

[4] If you don't know it already, look up the definition of *bounded distributive lattice* on the Web. Anticipating what is coming, think of "$a \leqslant b$" as saying "$a \to b$" or "$a \Rightarrow b$" .

Semantics of IPC: Heyting Algebras versus BA's

Based on the preceding and in the style of [LCHI, Def 2.4.2] :

- we can take a HA in the form

$$\mathcal{H} \triangleq \langle H, \sqcup, \sqcap, 0, 1, \Rightarrow, - \rangle$$

where unary "$-$" can be defined from "$\Rightarrow$", i.e. $-a \triangleq (a \Rightarrow 0)$, [5]

- and we can take a BA in the form

$$\mathcal{B} \triangleq \langle B, \sqcup, \sqcap, 0, 1, \Rightarrow, - \rangle$$

where binary "$\Rightarrow$" can be defined from "$-$", i.e. $(a \Rightarrow b) \triangleq (-a \sqcup b)$,

- and $\mathcal{H}$ becomes a BA if $(a \sqcup -a) = 1$ for every $a \in H$.

[5] We call "$-a$" the *pseudocomplement* of a because $a \sqcap -a = 0$ (why?) but not necessarily $a \sqcup -a = 1$. Otherwise, if $a \sqcup -a = 1$ for every $a \in H$, it will turn $\mathcal{H}$ into a Boolean algebra.

Semantics of IPC: Heyting Algebras

FACT: *The following expressions (and many others) are each equal to* 1 *in* $\mathcal{H}$ *:* [6]

1. $a \sqcap (a \Rightarrow b) \Rightarrow b$
2. $a \Rightarrow b \Rightarrow a \sqcap b$
3. $(a \sqcap b \Rightarrow a) \sqcap (a \sqcap b \Rightarrow b)$
4. $(a \Rightarrow b \Rightarrow c) \Rightarrow (a \Rightarrow b) \Rightarrow (a \Rightarrow c)$
5. $a \Rightarrow b \Rightarrow a$

for all $a, b, c \in H$, whereas $(a \sqcup -a)$ is not necessarily equal to 1, unless $\mathcal{H}$ is a BA.

Remark: In anticipation of the Curry-Howard Isomorphism , note the following:

Part 1 mimics the proof rule $(\to\mathrm{E})$,

Part 2 mimics the proof rule $(\wedge\mathrm{I})$,

Part 3 mimics the proof rule $(\wedge\mathrm{E})$,

Part 4 mimics the λ-term **S** ,

Part 5 mimics the λ-term **K** .

[6] By convention, '$\Rightarrow$' associates to the right, and '$\sqcap$' and '$\sqcup$' bind more tightly than '$\Rightarrow$' .

Semantics of IPC: Heyting Algebras

Expressions that are equal to 1 in HA's mimic *intuitionistically deducible wff's* :

- Relating algebraic expressions over signature $\{\sqcup, \sqcap, 0, 1, \Rightarrow, -\}$ and wff's:

$$\begin{array}{lll} \sqcup \text{ and } \sqcap & \text{interpret the connectives} & \wedge \text{ and } \vee, \\ 0 \text{ and } 1 & \text{interpret the constant symbols} & \bot \text{ and } \top, \\ \Rightarrow \text{ and } - & \text{interpret the connectives} & \to \text{ and } \neg. \end{array}$$

  By using the *algebraic operations* (on the left) interchangeably and at will with the *logical connectives and constant symbols* (on the right), we can write *informally*:

  FACT:

$$\begin{array}{llll} \varphi = 1 & \text{in all Heyting algebras} & \text{iff} & \vdash_{\mathsf{IPC}} \varphi \quad (\text{iff } \vdash_{\mathsf{IPC}} \varphi \leftrightarrow \top), \\ \varphi = \psi & \text{in all Heyting algebras} & \text{iff} & \vdash_{\mathsf{IPC}} \varphi \leftrightarrow \psi, \\ \varphi \leqslant \psi & \text{in all Heyting algebras} & \text{iff} & \vdash_{\mathsf{IPC}} \varphi \to \psi, \end{array}$$

  and many other equivalences between *algebraic equalities* in HA's (on the left) and *intuitionistically deducible wff's* (on the right) .

- Formal details and more in [LCHI, Sect 2.4, pp 39-41] .

Semantics of IPC: Heyting Algebras

- **THEOREM** (Soundness and Completeness for Heyting Algebras)
  For every set Γ of propositional wff's and every propositional wff φ, it holds that:

  $$\Gamma \vdash_{\mathsf{IPC}} \varphi \quad \text{iff} \quad \Gamma \models_{\mathsf{HA}} \varphi$$

  REMARK: We write "$\Gamma \models_{\mathsf{HA}} \varphi$" to mean that Γ semantically entails φ in *every* HA, not in just one HA, *i.e.* for every HA $\mathcal{H}$, if $\mathcal{H} \models \Gamma$ then $\mathcal{H} \models \varphi$. [7]

- Proof and details of the theorem are in [LCHI, Thm 2.4.7, pp 40-41] .

- Concrete examples of familiar HA's, connections with topological spaces and classical logic, are further examined in [LCHI, Sect 2.4, pp 41-43] .

[7] Note the contrast with CPC, for which we have the equivalence $\Gamma \vdash_{\mathsf{CPC}} \varphi$ iff $\Gamma \models_{\mathbb{B}} \varphi$, where it suffices to use only one BA, the standard two-valued $\mathbb{B}$. Is it possible to have $\Gamma \vdash_{\mathsf{IPC}} \varphi$ iff $\Gamma \models_{\mathcal{H}} \varphi$ for a single HA $\mathcal{H}$? Yes, it is possible, but then $\mathcal{H}$ must be infinite; see [LCHI, Thm 2.4.11, p 42] and the paragraph preceding it.

Alternative Semantics of IPC: Kripke Models (KM's)

- A ***Kripke Model*** is a triple of the form $\mathcal{C} \triangleq \langle C, \leqslant, \Vdash \rangle$ where C is a non-empty set of *states*, $\leqslant$ is a partial order on C, and $\Vdash \subseteq C \times \mathrm{PV}$, such that:

  $$\text{if} \;\; c \leqslant c' \;\; \text{and} \;\; c \Vdash p \;\; \text{then} \;\; c' \Vdash p$$

  The relation $\Vdash$ (read "*forces*") is extended to $C \times \Phi$:

  $$\begin{array}{lll} c \Vdash \varphi \vee \psi & \text{iff} & c \Vdash \varphi \text{ or } c \Vdash \psi \\ c \Vdash \varphi \wedge \psi & \text{iff} & c \Vdash \varphi \text{ and } c \Vdash \psi \\ c \Vdash \varphi \to \psi & \text{iff} & c' \Vdash \psi \text{ for every } c' \text{ such that } c \leqslant c' \text{ and } c' \Vdash \varphi \\ c \nVdash \bot & & \end{array}$$

  The preceding implies:

  $$\begin{array}{l} c \Vdash \top \\ c \Vdash \neg\varphi \quad \text{iff} \quad c' \nVdash \varphi \text{ for every } c' \text{ such that } c \leqslant c'. \end{array}$$

- Further details and examples in [LCHI, Sect 2.5, pp 43-44].

Alternative Semantics of IPC: Kripke Models (KM's)

- From the preceding definition, several easy properties follow, including:
  - **FACT** (Persistence)
    If $c \leqslant c'$ and $c \Vdash \varphi$ then $c' \Vdash \varphi$.
  - **FACT** (Density)
    $c \Vdash \neg\neg\varphi$ iff for every c_1 s.t. $c \leqslant c_1$ there is c_2 s.t. $c_1 \leqslant c_2$ and $c_2 \Vdash \varphi$.
- Every *Heyting algebra* can be turned into a *Kripke model* ,
  non-trivial details in [LCHI, Sect 2.5, p 45] .
- **THEOREM** (Soundness and Completeness for Kripke Models)
  For every set Γ of propositional wff's and every propositional wff φ, it holds that:

  $$\Gamma \vdash_{\mathrm{IPC}} \varphi \quad \text{iff} \quad \Gamma \models_{\mathrm{HA}} \varphi \quad \text{iff} \quad \Gamma \Vdash_{\mathrm{KM}} \varphi$$

  where we write "$\Gamma \Vdash_{\mathrm{KM}} \varphi$" to mean:
  for every KM $\mathcal{C} = \langle \mathcal{C}, \leqslant, \Vdash \rangle$ and every $c \in C$, if $\mathcal{C}, c \Vdash_{KM} \Gamma$ then $\mathcal{C}, c \Vdash_{KM} \varphi$.

Implicational Fragment of IPC

- There are several fragments of IPC, notably *minimal propositional logic* and *implicational propositional logic*.

  IPC($\to$) = *implicational IPC*,
  IPC($\to, \wedge, \vee$) = *minimal IPC*,
  and *full IPC* = IPC($\to, \bot, \wedge, \vee$), as in [LCHI, p 32-33].

- We consider the fragment IPC($\to$) if only because it provides an exact fit with the basic *simply-typed λ-calculus $\lambda_\to$* (covered in [LCHI, Chapt 3]).

  Indeed, the *type-assignment rules of $\lambda_\to$* are just annotated versions of the *natural-deduction proof rules of* IPC($\to$).

- To match the rules of IPC($\to, \bot, \wedge, \vee$) [LCHI, Sect 4.5] extends $\lambda_\to$ with:
  - *pair* and *projections* to match the proof rules for $\wedge$,
  - *injections* and *disjoint sum* to match the proof rules for $\vee$.

- **THEOREM** (Soundness and Completeness for IPC($\to$)) [LCHI, Sect 2.6]

  For every set Γ of wff's in IPC($\to$) *and every wff φ in* IPC($\to$), *it holds that:*

  $$\Gamma \vdash_{\mathrm{IPC}(\to)} \varphi \quad \text{iff} \quad \Gamma \models_{\mathrm{HA}} \varphi \quad \text{iff} \quad \Gamma \Vdash_{\mathrm{KM}} \varphi$$

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